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Spatial Geometry and the Wu-Yang Ambiguity

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ABSTRACT

We display continuous families of SU(2) vector potentials $A^a_i(x)$ in 3 space dimensions which generate the same magnetic field $B^{ai}(x)$ (with det $B \neq 0$). These Wu-Yang families are obtained from the Einstein equation $R_{ij} = -2G_{ij}$ derived recently via a local map of the gauge field system into a spatial geometry with 2-tensor $G_{ij} = B^a{}_i B^a{}_j \det B$ and connection Γ^i_{jk} with torsion defined from gauge covariant derivatives of B.

The Wu-Yang ambiguity [1] is the phenomenon that two or more gauge inequivalent non-abelian potentials $A^a_i(x)$ generate the same field strength $F^a_{ij}(x)$. Although the original example is 3-dimensional, it was mainly the 4-dimensional case which was of past interest. Many examples of a discrete ambiguity have been exhibited, specifically two potentials A and $\bar{A}$ giving the same F (see [3] and references therein). The few examples of a continuous ambiguity were degenerate in some way: for example, they were effectively 2-dimensional. In this talk we summarize previously published work [2,3] and display examples of continuous families of potentials which generate the same magnetic field

$$B^{ai} = \epsilon^{ijk}\left[\partial_j A^a_k + \frac{1}{2}\epsilon^{abc}A^b_j A^c_k\right] \ . \tag{1}$$

In 3 dimensions there is no "algebraic obstruction" to an ambiguity. However, this fact is not sufficient to demonstrate that (1), viewed as a partial differential equation for A^b_j given B^{ai}, has multiple solutions, and it is this which we wish to explore here.

The 3-dimensional case is relevant for the Hamiltonian form of gauge field dynamics in $3+1$ dimensions and especially for an attempt [2] to transform from A^a_i to B^{ai} as the basic field variables. An intermediate step is to replace the A, B system by a set of gauge-invariant spatial geometric variables, namely a metric G_{ij} and connection Γ^k_{ij} with torsion. It turns out that the information we find on the Wu-Yang ambiguity invalidates the proposed form of Hamiltonian dynamics [2]. But

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the geometry is valid, and it is through the geometrical equations that our Wu-Yang information is obtained.

We begin with our first example. Consider the smooth, algebraically non-singular (*i.e.* det $B \neq 0$) magnetic field $B^{ai} = \delta^{ai}$, in Euclidean space with Cartesian coordinates x, y, z. It is easy to show explicitly that, for any real parameter β with $|\beta| > 1$, the 1-parameter family of potentials

$$A_i^a = \begin{pmatrix} \beta \pm \sqrt{\beta^2 - 1}\cos(z/\beta) & \pm\sqrt{\beta^2 - 1}\sin(z/\beta) & 0 \\ \pm\sqrt{\beta^2 - 1}\sin(z/\beta) & \beta \mp \sqrt{\beta^2 - 1}\cos(z/\beta) & 0 \\ 0 & 0 & 1/\beta \end{pmatrix} \tag{2}$$

all reproduce the same B^{ai}. Gauge inequivalence is demonstrated by the fact that the invariants $B^{aj} D_i B^{ak}$ depend on β and z. The particular magnetic field $B^{ai} = \delta^{ai}$ is invariant under rotations and translations of the configuration space $I\!R^3$ (the spatial rotations must be combined with a suitably chosen SO(3) gauge transformation, constant in this case, as is well known). Since (1) is also covariant, each such isometry which does not leave A_i^a invariant produces another Wu-Yang related potential. In this way one can extend the potentials displayed in (2) to a 4-parameter family, in which the wave has an arbitrary phase, $z \to z - z_0$ and direction $(0, 0, 1) \to \hat{k}$. (1) is covariant under diffeomorphisms. Thus examples of the Wu-Yang ambiguity automatically extend to entire orbits of the diffeomorphism group, and one can find a diffeomorphism under which the field $B^{ai} = \delta^{ai}$, which has infinite energy, transforms to a configuration $B'^{a\alpha}(y)$ which falls sufficiently fast as $y^\alpha \to \infty$ that it has finite energy.

Let us now review the spatial geometry which is the main tool used in this work. Let ${B^a}_i(x)$ denote the matrix inverse of the magnetic field of SU(2) gauge theory, and $\det B = \det B^{ai}$, which is gauge invariant. Then $G_{ij}(x) = {B^a}_i(x){B^a}_j(x) \det B$ is a gauge invariant symmetric tensor (under diffeomorphisms). The following geometry, which obviously uses the fact that the gauge group SU(2) is also the tangent space group of a 3-manifold, emerged from the physical aim of studying the action of the electric field on gauge invariant state functionals $\psi[G]$.

The quantity $b_i^a = |\det B|^{\frac{1}{2}} B_i^a$ is essentially a frame for G_{ij}. One may apply a Yang-Mills covariant derivative and define a quantity Γ_{ij}^k as follows:

$$D_i b_j^a = \Gamma_{ij}^k b_k^a. \tag{3}$$

It can be shown that Γ is a metric compatible connection for G, and can be written as [2]

$$\Gamma_{ij}^k = \dot{\Gamma}_{ij}^k(G) - {K_{ij}}^k \tag{4}$$

where $\dot{\Gamma}$ is the Christoffel connection and K is the contortion tensor, which is antisymmetric in the last pair of indices, $K_{ijk} = -K_{ikj}$. Further manipulation of (3) leads to

$$R_{ij}(\Gamma) = -2G_{ij} \tag{5}$$

which defines an Einstein geometry with torsion. One may show using the second Bianchi identity of curvatures with torsion, that an integrability condition for (5)

is that the contortion tensor is traceless, ${K_{kj}}^k = 0$, and that this is also a direct requirement of the gauge field Bianchi identity, $D_i B^{ai} = 0$, applied to the definition (3) of Γ.

The discussion above defines the forward map from Yang-Mills fields A and B, always related by (1), to geometric variables G and Γ defined by explicit local formulas above. The gauge field Ricci and Bianchi identities then imply that G and Γ are related by the Einstein condition (5) with traceless contortion. The fundamental reason for the Einstein geometry is that the magnetic field is simultaneously the curvature (1) of the gauge connection A and also essentially the frame of the spatial geometry.

One may also ask about the inverse map from tensor $G_{ij}(x)$ and connection $\Gamma^k_{ij}(x)$ on $I\!R^3$ to gauge fields. Suppose that a frame b^a_i, with $\det b > 0$, is constructed for G_{ij} by any standard method, then (3) can be written out as

$$\partial_i b^a_j - \Gamma^k_{ij} b^a_k + \epsilon^{abc} A^b_i b^c_j = 0 \ . \tag{6}$$

This is just the "dreibein postulate" with A essentially the spin connection, and one can solve for A, obtaining

$$A^a_i = -\frac{1}{2}\epsilon^{abc} b^{bj} \left(\partial_i b^c_j - \Gamma^k_{ij} b^c_k\right) \tag{7}$$

while the magnetic field is defined from the inverse frame by

$$B^{ai}(x) = |\det G_{jk}|^{\frac{1}{2}} b^{ai}. \tag{8}$$

Thus given a frame one obtains the magnetic field from (8), while both b and Γ are required to define the potential via (7). Since the frame is unique up to a local $SO(3)$ rotation, these maps define A and B uniquely up to an $SU(2)$ gauge transformation. Furthermore, A and B defined in this way satisfy the gauge theory relation (1) if Γ and G satisfy (5).

Thus the gauge theory Wu-Yang ambiguity will appear whenever the Einstein equation (5) viewed as a partial differential equation for K, given G, has multiple solutions. To investigate this it is useful to use the representation

$$K^i{}_{jk} = \epsilon_{jkn} S^{ni} \frac{1}{|\det G|^{1/2}} \tag{9}$$

which automatically satisfies the antisymmetry and tracelessness requirements if S^{ni} is a symmetric tensor. When (5) is expanded out using (4) and (9), one finds that the Einstein equation is equivalent to

$$\frac{\epsilon^{jkl}}{|\det G|^{1/2}} \dot{\nabla}_k S_{li} - \left(S^j_k S^k_i - S^k_k S^j_i\right) = \dot{R}^j_i + 2\delta^j_i. \tag{10}$$

In (10) $\dot{\nabla}_k$ indicates a spatial covariant derivative with Christoffel connection $\dot{\Gamma}$ and $\dot{R}_{ij}$ is the conventional symmetric Ricci tensor. The $\epsilon\dot{\nabla}S$ term is non-symmetric, so that (10) comprises 9 equations for the 6 components of S_{ij}. However it was shown

explicitly [2] that there is a Bianchi identity which imposes 3 constraints on the 9 equations, so there is no reason to think that (10) is an overdetermined system. From (4), (7) and (9), A can be expressed in terms of S as

$$A_i^a = -\frac{1}{2}\epsilon^{abc}b^{bj}\dot{\nabla}_i b_j^c - b^{ak}S_{ki}. \tag{11}$$

Our approach to the Wu-Yang ambiguity is to take an input metric $G_{ij}(x)$ and study the solutions of (10) for the torsion $S_{ij}(x)$. It is not clear why this should be a simpler method than to study directly whether (1) has multiple solutions for A, given B. Perhaps it is because an equation for the 6 components of S_{ij} is simpler to handle than an equation for the 9 components of A_i^a, but it may just be an historical accident that has led us to approach the Wu-Yang ambiguity via the spatial geometry.

Before beginning to study applications of (10), it is perhaps useful to note that (3) indicates that Γ is completely determined by first covariant derivatives D_iB^{aj} of the magnetic field. It then follows from properties of the inverse map discussed above that there is no Wu-Yang ambiguity for the potential A_i^a, if we require that both B and DB are preserved*. In 4 dimensional $SU(2)$ gauge theory, the field strength and its first two covariant derivatives determine the potential locally uniquely.

The Wu-Yang ambiguity indicates that the potential $A_i^a(x)$ contains gauge invariant information beyond that in the magnetic field $B^{ai}(x)$. Therefore the change of field variable $A_i^a \to B^{ai}$ which was the basis of the version of gauge invariant Hamiltonian dynamics presented in [2] is invalid. The discrete 2:1 ambiguity envisaged there could be handled, but it is probably impossible to deal with a continuous ambiguity without serious revision of the proposal.

It is the tensor $G_{ij}(x) = \delta_{ij}$ that corresponds to the magnetic field $B^{ai} = \delta^{ai}$, and it can be seen without difficulties that the torsion solutions of (10) are related in this simple case to the potentials of (2) by $A_i^a(z) = -S_{ai}(z)$. Note that at $\beta = 1$, the solution (2) reduces to $S_{ij} = \delta_{ij}$. We found the family of solutions (2) by first linearizing about $S_{ij} = \delta_{ij}$ and using Fourier analysis to find linearized modes of wave number $k^2 = 1$. This led us to investigate the single variable ansatz $S_{ij}(z)$ which reduces (10) to a non linear system of ordinary differential equations. Some fiddling then led to (2), which is unique within this ansatz (except for translation $z \to z - z_0$). One can show that the only spherically symmetric solutions of (10) for input $G_{ij} = \delta_{ij}$ are the solutions $S_{ij} = \pm\delta_{ij}$. There is a heuristic argument that the potentials displayed in (2) together with those obtained from them by translation and rotation are the only potentials for the field $B^{ai} = \delta^{ai}$ which continuously limit to potentials $\bar{A}_i^a = \delta_i^a$ with $\beta = 1$ in (2). The reason is that one can show using the Fourier transform that the set of linear perturbations about $\bar{A}_i^a$ obtained in the $\beta \to 1$ limit of our Wu-Yang family are complete.

* Actually it is sufficient to require that $D_{[i}b_{j]}^a$ is preserved, because this determines the torsion tensor from (3).

Another set of Wu-Yang examples emerges from the 3-dimensional hyperbolic metrics

$$ds^2 = \frac{1}{c^2 z^2}\left(dx^2 + dy^2 + dz^2\right) \tag{12}$$

for which $R_{ij} = -2c^2 G_{ij}$. One may anticipate that the case $c^2 = 1$ is especially simple because the right side of (10) vanishes. It turns out that one can also make the $\epsilon\dot{\nabla}S$ and SS terms vanish separately. There is no integrability constraint when $\dot{\nabla}_j$ is applied to the former condition, while the second condition implies that S_{ij} is a rank 1 dyadic matrix. With this structure in view one can easily find that within the two variable ansatz $S_{ij}(z,x)$, there is the family of solutions

$$S_{ij}(z,x) = \delta_{i1}\delta_{j1}\frac{1}{z}h(x) \tag{13}$$

which involves an arbitrary function of the variable x. The solution can be rotated by an angle θ in the x,y plane to obtain

$$\begin{aligned} S_{ij} &= \frac{1}{z}h(x\cos\theta + y\sin\theta)V_i V_j, \\ V_i &= (\cos\theta, \sin\theta, 0). \end{aligned} \tag{14}$$

We have not studied the application of the full $SO(2,1)\times SO(2,1)$ isometry group of the metric (12), but more solutions seem likely. In this frame the magnetic field is given simply by $B^{ai} = \delta^{ai}/z^2$ while the gauge potential corresponding to (13) is obtained from (11):

$$A_1^1 = -h(x), \quad A_2^1 = -A_1^2 = 1/z, \tag{15}$$

with the rest of the components vanishing. In this frame, the magnetic field B^{ai} is singular on the plane $z = 0$. It is straightforward to transform our configuration to a frame in which both the magnetic field and gauge potential are manifestly regular over all of $\mathbb{R}^3$ (see [3]).

When $c^2 \neq 1$ the full nonlinear equations are very difficult to handle, so we restrict ourselves to a perturbative expansion about the symmetric solution $\bar{S}_{ij} = \sqrt{1-c^2}G_{ij}$ of (10) by setting $S_{ij} = \bar{S}_{ij} + \hat{\Sigma}_{ij}$. The perturbation $\hat{\Sigma}_{ij}$ satisfies the linear equation

$$(cz)\epsilon^{jkl}\dot{\nabla}_k\hat{\Sigma}_{li} + \sqrt{1-c^2}(\hat{\Sigma}_{ji} + \hat{\Sigma}_{kk}\delta_{ji}) = 0. \tag{16}$$

(the placement of the j index reflects the removal of the conformal factor.) The 9 equations for the 6 components of $\hat{\Sigma}$ cannot all be independent, and the fact that we find a consistent solution below is a practical test of the exact Bianchi identity satisfied by (10). Note that the ij contraction of (16) immediately tells us that the trace $\hat{\Sigma}_{kk} = 0$.

Because of the x-translation symmetry of the metric (12) we look for a solution of the form $\hat{\Sigma}_{ij}(z,x) = \Sigma_{ij}(z,k)e^{ikx}$. The 9 equations of (16) can be manipulated to obtain a second order differential equation for the component Σ_{23}

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - k^2 z^2 + \frac{1-c^2}{c^2}\right)\Sigma_{23} = 0, \tag{17}$$

while the other components are related to Σ_{23} by

$$\begin{aligned}
\Sigma_{33} &= \frac{-ikc}{\sqrt{1-c^2}} z\Sigma_{23}, \\
\Sigma_{13} &= \frac{c}{\sqrt{1-c^2}} z\frac{d}{dz}\Sigma_{23}, \\
\Sigma_{12} &= \frac{i}{k}(\frac{d}{dz} - \frac{1}{z})\Sigma_{23}, \\
\Sigma_{11} &= \frac{-ic}{k\sqrt{1-c^2}} z\frac{d^2}{dz^2}\Sigma_{23}, \\
\Sigma_{22} &= -(\Sigma_{11} + \Sigma_{33}).
\end{aligned} \tag{18}$$

Note that (20) is the differential equation for Bessel functions of imaginary argument ikz and index $p = \sqrt{(c^2-1)/c^2}$ which is also imaginary when $c^2 < 1$ and the symmetric torsion $\bar{S}_{ij}$ is real.

Note that the wave number k of the linear perturbation is not restricted in contradistinction to the flat metric where, as can be seen from the small amplitude limit $\beta \to 1$ in (2), the wave number $k = 1$ is required. This means that the general real superposition

$$\int dk\varphi(k)\Sigma_{ij}(kz)e^{ikx} + \text{c.c.} \tag{19}$$

is also a solution, so we have the freedom of an arbitrary function at the linear level. We expect that (19) can be used as the "input" to the system of differential equations determining second and higher order perturbative solutions of (10), and that the functional freedom of $\varphi(k)$ remains. Thus the qualitative picture of the torsion solutions for the hyperbolic metrics for all values of c is that they contain an arbitrary function of a single variable and the additional parametric freedom obtained from isometries. The case $c = 1$ is special only because exact solutions can be easily obtained. A similar analysis to the above follows for the case of $2+1$ product metrics, in which functional freedom is again expected to persist in higher order perturbative solutions.

In summary, what we have discussed in this talk are several examples of a continuous Wu-Yang ambiguity for $SU(2)$ gauge fields in 3 dimensions and a new technique, namely the Einstein space condition (10) for obtaining such field configurations. It is intriguing to ask about the systematics of the ambiguity; namely what properties of the B-field determine the degree of ambiguity in the associated potentials A. Our examples provide at least a limited view of this systematics. Certainly an ambiguity is generated whenever there is a symmetry transformation of B which acts nontrivially on A, but this is not enough to explain the free parameter β in (2), nor the arbitrary functions such as $F(x)$ in the example (13-15) or in the linear solutions. Gauge field topology does not seem to be the issue here for two reasons. First of all the ambiguity can be exhibited in any compact subset of the configuration space $\mathbb{R}^3$. Second, if we are given in some gauge a Wu-Yang family with suitable behavior at spatial infinity one can apply a gauge transformation to change the topological class at will. Of course one does expect that, except for singularities of the map (1),

the degree of ambiguity in A will not change as the parameters of B are smoothly varied. Our examples appear to be consistent with this requirement, although the discrete ambiguity found when $\dot{R}_{ij} = 0$ must be understood as a singular limit of the case of non-zero curvature. It is interesting that the Riemannian curvature $\dot{R}_{ij}$ of the metric G_{ij} obtained from B plays a role both in the ease of obtaining solutions for A and in the qualitative nature of the ambiguity.

References

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